

Unified $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator algebra: irreducible representations and induced deformed harmonic oscillator

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Abstract

A new deformed canonical commutation relation, generalizing various known deformations, is defined together with its structure function of deformation. Then, the related irreducible representations are characterized and classified. Finally, the discrete spectrum of the corresponding deformed harmonic oscillator Hamiltonian is investigated and discussed.

Key-words: Deformed oscillator algebra, algebra representations, discrete spectrum, structure function.

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1 Introduction

In recent years, a lot of interest has been devoted to the study of various quantum deformations of bosonic oscillators. From the mathematical point of view, this popularity is due to their connection with the non-commutative geometry, special functions of multiparameter analysis [1] and other fields of mathematics. From the other side, there are some hopes that the deformed oscillator can be more suitable for physical studies of nonlinear phenomena than the usual bosonic oscillator of the standard quantum mechanics. Such hopes are supported by several applications in conformal field theory [2], in nuclear spectroscopy [3, 4], in nonlinear quantum optics, in condensed matter physics [5], and in the description of systems with non-standard statistics and energy spectrum [6].

Deformations of the oscillator arose from the successive generalizations of the Arik-Coon [7] and Biedenharm-Macfarlane [8, 9] q -oscillators. As in the classical case, the problem of realization of q -deformed algebras by the one-parameter deformed creation and annihilation operators is important for the representation theory of quantum groups. Various attempts have been made to introduce new parameters in the rich and varied choices of deformed commutation relations. For instance, the study of two-parameter quantum groups, mainly based on the famous (p, q) -deformation, and their representations has started in the works [10, 11]. It is worth noticing that the increase of the number of deformation parameters makes the method of the deformations more flexible.

In this work, we consider the generalized $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator algebra as a generalization of the $(q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator algebra recently defined in [12], investigate in detail its representations which are suitably classified, and scrutinize the properties of the corresponding deformed harmonic oscillator.

The paper is organized as follows. In section 2, we recall some multiparameter deformations which appear in the literature and define our algebra. In section 3, we study some relevant properties of this algebra. In section 4, we give a classification of the representations of this algebra. We discuss properties of discrete spectrum of the Hamiltonian of the deformed harmonic oscillator corresponding to this system and end with some concluding remarks in section 5.

2 Multiparameter deformations of the Heisenberg-Weyl algebra: overview and new $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator algebra

The nondeformed oscillator algebra of the quantum harmonic oscillator is defined by the canonical commutation relations

$$[a, a^\dagger] = I, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger. \quad (2.1)$$

The common used deformation of this Heisenberg-Weyl algebra is defined as the algebra generated by the set of operators $\{I, a, a^\dagger, N\}$ and the relations [13]

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger, \quad a^\dagger a = f(N), \quad aa^\dagger = f(N+1) \quad (2.2)$$

where the structure function of the deformation, f , is positive and analytic.

The structure function $f(x)$ characterizes the deformation scheme. In our context, let us briefly recall some multiparameter deformations known in the literature. Other multiparameter generalizations of one-parameter deformations were enumerated in [12].

- (i) The (p, q) -generalization of the algebra introduced by Chakrabarty and Jagannathan in [14]. It is generated by the operators I, a, a^\dagger and N as follows:

$$\begin{aligned} aa^\dagger - qa^\dagger a &= p^{-N}, & aa^\dagger - p^{-1}a^\dagger a &= q^N \\ [N, a] &= -a, & [N, a^\dagger] &= a^\dagger \\ f(n) &= \frac{p^{-n} - q^n}{p^{-1} - q}. \end{aligned} \quad (2.3)$$

- (ii) The (p, q) -generalization of the Quesne's deformed bosonic algebra defined as [15]:

$$\begin{aligned} p^{-1}aa^\dagger - a^\dagger a &= q^{-N-1}, & qaa^\dagger - a^\dagger a &= p^{N+1} \\ [N, a] &= -a, & [N, a^\dagger] &= a^\dagger \\ f(n) &= \frac{p^n - q^{-n}}{q - p^{-1}}. \end{aligned} \quad (2.4)$$

- (iii) The $(p, q; \alpha, \beta, l)$ -deformed oscillator algebra, given by the generators I, a, a^\dagger, N and the commutation relations [16]

$$\begin{aligned} aa^\dagger - q^l a^\dagger a &= p^{-\alpha N - \beta}, & aa^\dagger - p^{-l} a^\dagger a &= q^{\alpha N + \beta} \\ [N, a] &= -la, & [N, a^\dagger] &= la^\dagger \\ f(n) &= \frac{p^{-\alpha n - \beta} - q^{\alpha n + \beta}}{p^{-l} - q^l} \end{aligned} \quad (2.5)$$

with $\alpha, \beta, l \in \mathbb{R}$.

In this work, we define the $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator algebra as:

$$\begin{aligned} aa^\dagger - p^\nu a^\dagger a &= (1 + 2\gamma K)q^{\alpha N + \beta}, & [N, a] &= -a, & [N, a^\dagger] &= a^\dagger \\ Ka &= -aK, & Ka^\dagger &= -a^\dagger K, & [N, K] &= 0, & N^\dagger &= N, & K^\dagger &= K \end{aligned} \quad (2.6)$$

where $p, q \in \mathbb{R}_+$, $\alpha, \beta, \nu, \gamma \in \mathbb{R}$. Particular algebras are readily recovered. For instance, in the limit $p \rightarrow q$, one finds the $(q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator algebra introduced by Burban [12]. Furthermore, the algebra (2.6) is also a generalization of the above mentioned (p, q) -algebras.

3 Unified $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator algebra: structure function and pertinent properties

The generalized $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator and its deformed Heisenberg-Weyl algebra are defined by the positive structure function f , satisfying $f(0) = 0$. The Fock realization of this algebra covers with the form

$$\begin{aligned} a|n\rangle &= \sqrt{f(n)}|n-1\rangle, & a^\dagger|n\rangle &= \sqrt{f(n+1)}|n+1\rangle \\ N|n\rangle &= n|n\rangle, & K|n\rangle &= (-1)^n|n\rangle \end{aligned} \quad (3.1)$$

with $n \in \mathbb{N}$.

Proposition 3.1

The structure function of the $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator algebra (2.6) is given by

$$f(n) = \begin{cases} q^\beta \left(\frac{p^{\nu} - q^{\alpha}}{p^{\nu} - q^{\alpha}} + 2\gamma \frac{p^{\nu} - (-1)^n q^{\alpha}}{p^{\nu} + q^{\alpha}} \right), & \text{if } p^{\nu} \neq q^{\alpha} \\ \left(n + 2\gamma \frac{1 - (-1)^n}{2} \right) \times q^{(n-1)\alpha + \beta}, & \text{if } p^{\nu} = q^{\alpha}. \end{cases} \quad (3.2)$$

Proof. By applying the basis $|n\rangle$ on the relation $aa^\dagger - p^{\nu}a^\dagger a = (1 + 2\gamma K)q^{\alpha N + \beta}$, we obtain the recurrence relation

$$f(n+1) - p^{\nu}f(n) = (1 + 2\gamma(-1)^n)q^{\alpha n + \beta} \quad (3.3)$$

whose solution is

$$f(n) = \sum_{k=0}^{n-1} p^{(n-k-1)\nu} q^{\alpha k + \beta} + 2\gamma \sum_{k=0}^{n-1} (-1)^k p^{(n-k-1)\nu} q^{\alpha k + \beta}. \quad (3.4)$$

Indeed, for $n = 1$, $f(1) = q^{\alpha \times 0 + \beta} + 2\gamma(-1)^0 q^{\alpha \times 0 + \beta}$. Now, we suppose that the function $f(k)$ is given by (3.4) for $k \leq n$. Then we have

$$\begin{aligned} f(n+1) &= p^{\nu}f(n) + (1 + 2\gamma(-1)^n)q^{\alpha n + \beta} \\ &= p^{\nu} \left(\sum_{k=0}^{n-1} p^{(n-k-1)\nu} q^{\alpha k + \beta} + 2\gamma \sum_{k=0}^{n-1} (-1)^k p^{(n-k-1)\nu} q^{\alpha k + \beta} \right) \\ &\quad + (1 + 2\gamma(-1)^n)q^{\alpha n + \beta} \\ &= \sum_{k=0}^n p^{(n-k)\nu} q^{\alpha k + \beta} + 2\gamma \sum_{k=0}^n (-1)^k p^{(n-k)\nu} q^{\alpha k + \beta} \end{aligned}$$

which proves the claim.

Hence, if $p^{\nu} \neq q^{\alpha}$

$$f(n) = p^{(n-1)\nu} q^{\beta} \left(\frac{1 - (p^{-\nu} q^{\alpha})^n}{1 - p^{-\nu} q^{\alpha}} + 2\gamma \frac{1 - (-1)^n (p^{-\nu} q^{\alpha})^n}{1 + p^{-\nu} q^{\alpha}} \right)$$

$$= q^\beta \left(\frac{p^{n\nu} - q^{n\alpha}}{p^\nu - q^\alpha} + 2\gamma \frac{p^{n\nu} - (-1)^n q^{n\alpha}}{p^\nu + q^\alpha} \right)$$

and

$$\begin{aligned} f(n) &= \left(\sum_{k=0}^{n-1} 1^k + 2\gamma \sum_{k=0}^{n-1} (-1)^k \right) \times q^{(n-1)\alpha+\beta} \\ &= \left(n + 2\gamma \frac{1 - (-1)^n}{2} \right) \times q^{(n-1)\alpha+\beta} \end{aligned}$$

if $p^\nu = q^\alpha$. \square

Let us study the positivity of the $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed structure function $f(n)$. The inequality $f(n) > 0$ can be rewritten as

$$(p^{n\nu} - q^{n\alpha}) \left\{ \frac{1}{p^\nu - q^\alpha} + 2\gamma \frac{1}{p^\nu + q^\alpha} \right\} > 0 \quad (3.5)$$

if n is even, and

$$\frac{p^{n\nu} - q^{n\alpha}}{p^{n\nu} + q^{n\alpha}} \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha} + 2\gamma > 0 \quad (3.6)$$

if n is odd.

Let us define the sequence

$$u_n = \begin{cases} u(\alpha, \nu) \frac{p^{n\nu} - q^{n\alpha}}{p^{n\nu} + q^{n\alpha}}, & \text{if } n > 1 \\ 1, & \text{if } n = 1 \end{cases} \quad (3.7)$$

where $u(\alpha, \nu) = \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$.

If

$$f(x) = u(\alpha, \nu) \frac{p^{x\nu} - q^{x\alpha}}{p^{x\nu} + q^{x\alpha}}$$

we have

$$\frac{df(x)}{dx} = f'(x) = \frac{(\nu \ln p - \alpha \ln q)}{p^\nu - q^\alpha} \times \frac{2(p^\nu + q^\alpha)}{(p^{x\nu} + q^{x\alpha})^2}.$$

Since $(\nu \ln p - \alpha \ln q)(p^\nu - q^\alpha) > 0$, we deduce that $(u_n)_{n \geq 1}$ is an increasing sequence. We have $u_n > 1$ for all $n \in \mathbb{N} \setminus \{0\}$. To have $u_n + 2\gamma > 0$, $\forall n \in \mathbb{N} \setminus \{0\}$, it is sufficient that $1 + 2\gamma > 0$.

The relation (3.5) is equivalent to $-2\gamma < \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$ if $\nu \ln p > \alpha \ln q$ and $2\gamma < -\frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$ if $\nu \ln p < \alpha \ln q$.

Therefore we obtain:

Proposition 3.2 *The structure function $f(n)$ is positive for*

$$-2\gamma < 1 \quad \text{if} \quad \nu \ln p > \alpha \ln q \quad (3.8)$$

and

$$-1 < 2\gamma < -\frac{p^\nu + q^\alpha}{p^\nu - q^\alpha} \quad \text{if} \quad \nu \ln p < \alpha \ln q. \quad (3.9)$$

Proposition 3.3 *From the relation $aa^\dagger - p^\nu a^\dagger a = (1 + 2\gamma K)q^{\alpha N + \beta}$, we get the formula*

$$a(a^\dagger)^n - p^{n\nu}(a^\dagger)^n a = [n; \alpha, \beta, \nu; \gamma K] (a^\dagger)^{n-1} q^{\alpha N + \beta}, \quad \forall n \in \mathbb{N} \setminus \{0\} \quad (3.10)$$

where

$$[n; \alpha, \beta, \nu; \gamma K] = \begin{cases} \frac{p^{n\nu} - q^{n\alpha}}{p^\nu - q^\alpha} + 2\gamma K \frac{q^{n\alpha} - (-1)^n p^{n\nu}}{p^\nu + q^\alpha}, & \text{if } p^\nu \neq q^\alpha \\ nq^{\alpha(n-1)} + 2\gamma K q^{\alpha(n-1)} \times \left(\frac{1 - (-1)^n}{2} \right), & \text{if } p^\nu = q^\alpha. \end{cases}$$

Proof.

Let us prove (3.10) by recurrence. It is obviously true for $n = 1$. For $n = 2$, we have

$$\begin{aligned} a(a^\dagger)^2 &= aa^\dagger a^\dagger \\ &= (p^\nu a^\dagger a + (1 + 2\gamma K)q^{\alpha N + \beta})a^\dagger \\ &= p^{2\nu}(a^\dagger)^2 a + p^\nu a^\dagger (1 + 2\gamma K)q^{\alpha N + \beta} + (1 + 2\gamma K)q^{\alpha N + \beta} a^\dagger \\ &= p^{2\nu}(a^\dagger)^2 a + p^\nu (1 - 2\gamma K)a^\dagger q^{\alpha N + \beta} + (1 + 2\gamma K)a^\dagger q^{\alpha(N+1) + \beta} \end{aligned}$$

so that

$$a(a^\dagger)^2 - p^{2\nu}(a^\dagger)^2 a = \{p^\nu + q^\alpha + 2\gamma K(q^\alpha - p^\nu)\} a^\dagger q^{\alpha N + \beta}.$$

This latter relation can be put in the form

$$a(a^\dagger)^2 - p^{2\nu}(a^\dagger)^2 a = \left\{ \frac{p^{2\nu} - q^{2\alpha}}{p^\nu - q^\alpha} + 2\gamma K \frac{q^{2\alpha} - (-1)^2 p^{2\nu}}{p^\nu + q^\alpha} \right\} a^\dagger q^{\alpha N + \beta} \quad (3.11)$$

if $p^\nu \neq q^\alpha$, and

$$a(a^\dagger)^2 - p^{2\nu}(a^\dagger)^2 a = \left\{ 2q^\alpha + 2\gamma K q^{\alpha(2-1)} \times \left(\frac{1 - (-1)^2}{2} \right) \right\} a^\dagger q^{\alpha N + \beta} \quad (3.12)$$

if $p^\nu = q^\alpha$.

We claim that the relation (3.10) is true for $k \leq n$. Then, we have

$$\begin{aligned} a(a^\dagger)^{n+1} &= a(a^\dagger)^n a^\dagger \\ &= \left\{ p^{n\nu}(a^\dagger)^n a + [n; \alpha, \beta, \nu; \gamma K] (a^\dagger)^{n-1} q^{\alpha N + \beta} \right\} a^\dagger \\ &= p^{n\nu}(a^\dagger)^n \left\{ p^\nu a^\dagger a + (1 + 2\gamma K)q^{\alpha N + \beta} \right\} \\ &\quad + [n; \alpha, \beta, \nu; \gamma K] (a^\dagger)^n q^{\alpha(N+1) + \beta} \\ &= p^{(n+1)\nu}(a^\dagger)^{n+1} a + p^{n\nu}(1 + 2(-1)^n \gamma K)(a^\dagger)^n q^{\alpha N + \beta} \\ &\quad + q^\alpha [n; \alpha, \beta, \nu; \gamma K] (a^\dagger)^n q^{\alpha N + \beta}. \end{aligned}$$

So,

$$\begin{aligned} a(a^\dagger)^{n+1} - p^{(n+1)\nu}(a^\dagger)^{n+1} a &= \left\{ p^{n\nu} + q^\alpha \left(\frac{p^{n\nu} - q^{n\alpha}}{p^\nu - q^\alpha} \right) \right. \\ &\quad \left. + 2\gamma K \left[(-1)^n p^{n\nu} + q^\alpha \frac{q^{n\alpha} - (-1)^n p^{n\nu}}{p^\nu + q^\alpha} \right] \right\} (a^\dagger)^n q^{\alpha N + \beta} \end{aligned}$$

$$= [n+1; \alpha, \beta, \nu; \gamma K] (a^\dagger)^n q^{\alpha N + \beta} \quad (3.13)$$

if $p^\nu \neq q^\alpha$ and

$$\begin{aligned} & a(a^\dagger)^{n+1} - p^{(n+1)\nu} (a^\dagger)^{n+1} a \\ &= \{q^{n\alpha}(1 + 2(-1)^n \gamma K) + q^\alpha [n; \alpha, \beta, \nu; \gamma K]\} (a^\dagger)^n q^{\alpha N + \beta} \\ &= \left\{ (n+1)q^{n\alpha} + 2\gamma K q^{n\alpha} \times \left(\frac{1 - (-1)^{n+1}}{2} \right) \right\} (a^\dagger)^n q^{\alpha N + \beta} \\ &= [n+1; \alpha, \beta, \nu; \gamma K] (a^\dagger)^n q^{\alpha N + \beta} \end{aligned} \quad (3.14)$$

if $p^\nu = q^\alpha$, which proves the claim. \square

We readily obtain the generated function for $[n; \alpha, \beta, \nu; \gamma K]$:

$$\sum_{n=0}^{+\infty} [n; \alpha, \beta, \nu; \gamma K] z^n = \begin{cases} \frac{z}{1-q^\alpha z} \left(\frac{1}{1-p^\nu z} + 2\gamma K \frac{1}{1+p^\nu z} \right), & \text{if } p^\nu \neq q^\alpha \\ z \left(\frac{1}{(1-q^\alpha z)^2} + 2\gamma K \frac{1}{1-q^{2\alpha} z^2} \right), & \text{if } p^\nu = q^\alpha. \end{cases} \quad (3.15)$$

4 Irreducible representations of the unified $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator: characterization and classification

Here we give, following the work by Rideau [17], the classification of the irreducible representations of the $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator algebra.

This algebra possesses the following Casimir operators

$$C_1 = K^2, \quad C_2 = K e^{i\pi N}, \quad C_3 = e^{2i\pi N}. \quad (4.1)$$

Let w be an eigenvalue of C_2 corresponding to a given representation. Then, $K = w e^{-i\pi N}$. Let ψ_0 be a common eigenvector of N and K :

$$N\psi_0 = \nu_0 \psi_0 \quad (4.2)$$

$$K\psi_0 = \gamma e^{-i\nu_0 \pi} \psi_0. \quad (4.3)$$

Due to the commutativity of $a^\dagger a$ and aa^\dagger with N and K , we may assume that:

$$aa^\dagger \psi_0 = \lambda_0 \psi_0 \quad (4.4)$$

$$a^\dagger a \psi_0 = \mu_0 \psi_0 \quad (4.5)$$

and $(\psi_0, \psi_0) = 1$. One can check that the vectors defined by

$$\phi_n = \begin{cases} (a^\dagger)^n \psi_0, & \text{if } n \geq 0 \\ a^{-n} \psi_0, & \text{if } n < 0 \end{cases} \quad (4.6)$$

are eigenvectors of $a^\dagger a$ and aa^\dagger :

$$a^\dagger a \phi_n = \lambda_n \phi_n \quad (4.7)$$

$$aa^\dagger \phi_n = \mu_n \phi_n. \quad (4.8)$$

Now, let us define the following vectors:

$$\psi_n = \begin{cases} \frac{1}{\sqrt{\prod_{k=1}^n \lambda_k}} (a^\dagger)^n \psi_0, & \text{for } n \geq 0 \\ \frac{1}{\sqrt{\prod_{k=1}^{-n} \lambda_{k+n}}} a^{-n} \psi_0, & \text{for } n < 0 \end{cases} \quad (4.9)$$

which are orthogonal states. The actions of basic operators on them are given by

$$a^\dagger \psi_n = \sqrt{\lambda_{n+1}} \psi_{n+1} \quad (4.10)$$

$$a \psi_n = \sqrt{\lambda_n} \psi_{n-1} \quad (4.11)$$

$$N \psi_n = (\nu_0 + n) \psi_n \quad (4.12)$$

$$K \psi_n = \frac{(-1)^n}{2\gamma} B \psi_n \quad (4.13)$$

where $B = 2\gamma w e^{-i\pi\nu_0} \in \mathbb{R}$.

The additional condition that we have to take into account is that λ_n and μ_{n-1} , being eigenvalues of nonnegative operators $a^\dagger a$ and aa^\dagger , respectively, should be nonnegative.

Applying the relation $aa^\dagger - p^\nu a^\dagger a = (1 + 2\gamma K)q^{\alpha N + \beta}$ on the vectors ψ_n yields $\lambda_{n+1} = p^\nu \lambda_n + q^{\alpha\nu_0 + \beta}(1 + (-1)^n B)q^{\alpha n}$ and one can prove by recurrence that

$$\lambda_n = \begin{cases} p^{n\nu} \lambda_0 + q^{\alpha\nu_0 + \beta} \times \left(\frac{p^{n\nu} - q^{n\alpha}}{p^\nu - q^\alpha} + B \frac{p^{n\nu} - (-1)^n q^{n\alpha}}{p^\nu + q^\alpha} \right), & \text{if } p^\nu \neq q^\alpha \\ q^{n\alpha} \lambda_0 + q^{\alpha\nu_0 + \beta} \times q^{(n-1)\alpha} \left[n + B \left(\frac{1 - (-1)^n}{2} \right) \right], & \text{if } p^\nu = q^\alpha. \end{cases} \quad (4.14)$$

(A) If $p^\nu = q^\alpha$, the condition $\lambda_n \geq 0$ is equivalent to

$$\lambda_0 \geq -q^{\alpha(\nu_0-1)+\beta} \left(n + B \frac{1 - (-1)^n}{2} \right). \quad (4.15)$$

Since

$$\lim_{n \rightarrow -\infty} -q^{\alpha(\nu_0-1)+\beta} \left(n + B \frac{1 - (-1)^n}{2} \right) = +\infty, \quad (4.16)$$

there exists an integer n_0 for which $\lambda_n \leq 0$ for $n \leq n_0$. Since $a^\dagger a \geq 0$, we have

$$a \psi_n = 0 \quad \text{for } n \leq n_0. \quad (4.17)$$

After possible renumbering, we may assume

$$a \psi_0 = 0, \quad \lambda_0 = 0. \quad (4.18)$$

Therefore, the representation is spanned by the vectors ψ_n , $n \geq 0$ and the eigenvalues λ_n are given by

$$\lambda_n = n q^{\alpha(\nu_0+n-1)+\beta} + B q^{\alpha(\nu_0+n-1)+\beta} \times \left(\frac{1 - (-1)^n}{2} \right). \quad (4.19)$$

The arbitrary values of the parameters ν_0 and $B \geq 0$ correspond to non-equivalent representations of (4.10)-(4.13).

(B) If $p^\nu \neq q^\alpha$, λ_n can be rewritten as

$$\lambda_n = p^{n\nu} \left\{ \lambda_0 + q^{\alpha\nu_0+\beta} \left[\frac{1}{p^\nu - q^\alpha} + \frac{B}{p^\nu + q^\alpha} - (p^{-\nu} q^\alpha)^n \left(\frac{1}{p^\nu - q^\alpha} + \frac{B(-1)^n}{p^\nu + q^\alpha} \right) \right] \right\} \quad (4.20)$$

so that its positivity is equivalent to

$$\lambda_0 q^{-(\alpha\nu_0+\beta)} + \frac{1}{p^\nu - q^\alpha} + \frac{B}{p^\nu + q^\alpha} \geq (p^{-\nu} q^\alpha)^{2k} \left(\frac{1}{p^\nu - q^\alpha} + \frac{B}{p^\nu + q^\alpha} \right)$$

and

$$\lambda_0 q^{-(\alpha\nu_0+\beta)} + \frac{1}{p^\nu - q^\alpha} + \frac{B}{p^\nu + q^\alpha} \geq (p^{-\nu} q^\alpha)^{2k+1} \left(\frac{1}{p^\nu - q^\alpha} - \frac{B}{p^\nu + q^\alpha} \right).$$

(B)₁ Assume that $\nu \ln p > \alpha \ln q$. Then, at least one of $\frac{1}{p^\nu - q^\alpha} \pm \frac{B}{p^\nu + q^\alpha}$ is positive. Therefore, there exists an integer n_0 such that for n odd/ or even $\lambda_n \leq 0$ for $n \leq n_0$, which implies $a\psi_n = 0$ for some $n \leq n_0$. After possible renumbering, we may assume

$$a\psi_0 = 0, \quad \lambda_0 = 0. \quad (4.21)$$

Therefore, the representations are given by

$$\lambda_n = q^{\alpha\nu_0+\beta} \times \left(\frac{p^{n\nu} - q^{n\alpha}}{p^\nu - q^\alpha} + B \frac{p^{n\nu} - (-1)^n q^{n\alpha}}{p^\nu + q^\alpha} \right). \quad (4.22)$$

Let us now study the positivity of λ_n .

If n is even,

$$\lambda_n = q^{\alpha\nu_0+\beta} (p^{n\nu} - q^{n\alpha}) \left(\frac{1}{p^\nu - q^\alpha} + \frac{B}{p^\nu + q^\alpha} \right) \quad (4.23)$$

and λ_n is positive when $B \geq -\frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$.

If n is odd, λ_n is positive if and only if

$$\frac{p^{n\nu} - q^{n\alpha}}{p^{n\nu} + q^{n\alpha}} \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha} + B \geq 0 \quad (4.24)$$

which is satisfied when $B \geq -1$.

We conclude that λ_n is positive for all $n \in \mathbb{N}$ if $B \geq -1$ since $-\frac{p^\nu + q^\alpha}{p^\nu - q^\alpha} < -1$.

For $B = -1$, $\lambda_1 = 0$. Therefore, we obtain

$$a = a^\dagger = 0; \quad N = \nu_0; \quad K = -\frac{1}{2\gamma}. \quad (4.25)$$

This representation is one-dimensional. For $B > -1$, the representation is spanned by the vectors ψ_n , $n \geq 0$. It is defined by (4.10)-(4.13) and (4.22) with $n \geq 0$.

(B)₂ Assume that $\nu \ln p < \alpha \ln q$ and one of the values $\frac{1}{p^\nu - q^\alpha} \pm \frac{B}{p^\nu + q^\alpha}$ is positive. In this case, there exists an integer n_0 such that for $n \geq n_0$, λ_n is negative for even or odd n_0 . This implies $a^\dagger \psi_n = 0$ for some $n \geq n_0$. After possible renumbering, we get $a^\dagger \psi_0 = 0$. From the relation $aa^\dagger \psi_0 = \mu_0 \psi_0$, we obtain $\mu_0 = 0$. Hence, $\lambda_0 = -p^{-\nu} q^{\alpha\nu_0 + \beta} (1 + B)$.

The condition $\lambda_0 \geq 0$ is equivalent to $B \leq -1$. It leads to

$$\lambda_n = q^{\alpha\nu_0 + \beta} \left\{ -p^{(n-1)\nu} (1 + B) + \frac{p^{n\nu} - q^{n\alpha}}{p^\nu - q^\alpha} + B \frac{p^{n\nu} - (-1)^n q^{n\alpha}}{p^\nu + q^\alpha} \right\} \quad (4.26)$$

for $n \in \mathbb{Z}_-$.

Let us now discuss the positivity of λ_n .

If n is odd, (4.26) can be rewritten as

$$\lambda_n = q^{\alpha\nu_0 + \beta} q^{n\alpha} (1 - (q^{-\alpha} p^\nu)^{n-1}) \left(\frac{1}{p^\nu - q^\alpha} - \frac{B}{p^\nu + q^\alpha} \right) \quad (4.27)$$

which is positive when $B \leq \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$.

Now, we suppose that $B \leq \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$. When n is even, we have

$$\lambda_n = q^{\alpha\nu_0 + \beta} \left\{ q^\alpha p^{(n-1)\nu} \left(\frac{1}{p^\nu - q^\alpha} - \frac{B}{p^\nu + q^\alpha} \right) - q^{n\alpha} \left(\frac{1}{p^\nu - q^\alpha} + \frac{B}{p^\nu + q^\alpha} \right) \right\} \quad (4.28)$$

which is always positive since $B < -\frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$.

We deduce that λ_n is positive for all $n \in \mathbb{Z}_-$ if $B \leq \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$.

For $B < \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$, the representation is given by (4.10)-(4.13) and (4.26) for $n \leq 0$.

For $B = \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$, we have $\lambda_n = 0$ if n is odd and $\lambda_n = \frac{2q^{\alpha(\nu_0 + n) + \beta}}{q^\alpha - p^\nu}$ if n is even. The representation is two-dimensional:

$$\begin{aligned} a\psi_0 &= \sqrt{\frac{2q^{\alpha\nu_0 + \beta}}{q^\alpha - p^\nu}} \psi_{-1}; & a\psi_{-1} &= 0 \\ a^\dagger \psi_0 &= 0; & a^\dagger \psi_{-1} &= \sqrt{\frac{2q^{\alpha\nu_0 + \beta}}{q^\alpha - p^\nu}} \psi_0 \\ N\psi_0 &= \nu_0 \psi_0; & N\psi_{-1} &= (\nu_0 - 1) \psi_{-1} \\ K\psi_0 &= \frac{1}{2\gamma} \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha} \psi_0; & K\psi_{-1} &= -\frac{1}{2\gamma} \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha} \psi_{-1}. \end{aligned} \quad (4.29)$$

(B)₃ Assume $\nu \ln p < \alpha \ln q$ and both values $\frac{1}{p^\nu - q^\alpha} \pm \frac{B}{p^\nu + q^\alpha}$ are non positive. Let us consider the following cases:

$$(a) : \lambda q^{-(\alpha\nu_0 + \beta)} + \frac{1}{p^\nu - q^\alpha} + \frac{B}{p^\nu + q^\alpha} > 0$$

We have $\lambda_n > 0$, for all $n \in \mathbb{Z}$ and the representation is given by (4.10)-(4.13) and (4.20) with $n \in \mathbb{Z}$.

$$(b) : \lambda q^{-(\alpha\nu_0+\beta)} + \frac{1}{p^\nu - q^\alpha} + \frac{B}{p^\nu + q^\alpha} = 0$$

In this case the condition $\lambda_n \geq 0$ for all $n \in \mathbb{Z}$ is equivalent to $|B| \leq -\frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$. If $|B| < -\frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$, the representation is given by (4.10)-(4.13), where

$$\lambda_n = -q^{\alpha\nu_0+\beta} q^{n\alpha} \left(\frac{1}{p^\nu - q^\alpha} + \frac{B(-1)^n}{p^\nu + q^\alpha} \right) \quad (4.30)$$

for all $n \in \mathbb{Z}$.

If $B = -\frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$, $\lambda_n = 0$ when n is even. Therefore, the representation is given by:

$$\begin{aligned} a^\dagger \psi_0 &= \sqrt{\frac{2q^{\alpha(\nu_0+1)+\beta}}{q^\alpha - p^\nu}} \psi_1; & a^\dagger \psi_1 &= 0 \\ a \psi_0 &= 0; & a \psi_1 &= \sqrt{\frac{2q^{\alpha(\nu_0+1)+\beta}}{q^\alpha - p^\nu}} \psi_0 \\ N \psi_0 &= \nu_0 \psi_0; & N \psi_1 &= (\nu_0 + 1) \psi_1 \\ K \psi_0 &= \frac{1}{2\gamma} \frac{p^\nu + q^\alpha}{q^\alpha - p^\nu} \psi_0; & K \psi_1 &= \frac{1}{2\gamma} \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha} \psi_1. \end{aligned} \quad (4.31)$$

If $B = \frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}$, $\lambda_n = 0$ when n is odd and $\lambda_n = \frac{2q^{\alpha(\nu_0+n)+\beta}}{q^\alpha - p^\nu}$ when n is even. Therefore, the representation is two-dimensional and is given by (4.29).

$$(c) : \lambda q^{-(\alpha\nu_0+\beta)} + \frac{1}{p^\nu - q^\alpha} + \frac{B}{p^\nu + q^\alpha} < 0$$

There exists n_0 such that $\lambda_n \leq 0$ for $n \leq n_0$, n even and odd. Therefore, the representation is given by (4.10)-(4.13) and (4.22) with $n \in \mathbb{N}$. To provide $\lambda_n \geq 0$ for all $n \geq 0$ we have to restrict B in the interval

$$-1 \leq B < -\frac{p^\nu + q^\alpha}{p^\nu - q^\alpha}. \quad (4.32)$$

For $B = -1$, the representation is given by (4.25).

5 $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator: Hamiltonian definition and energy spectrum computation

The Hamiltonian of the $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator, defined in an analogous way as that of the usual harmonic oscillator as:

$$\mathcal{H} = \frac{\hbar w}{2} (a a^\dagger + a^\dagger a), \quad (5.1)$$

can be rewritten in terms of the number operator N as:

$$\begin{aligned} \mathcal{H} &= \frac{\hbar w}{2} (f(N) + f(N+1)) \\ &= \frac{\hbar w}{2} q^\beta \left\{ \frac{p^{\nu N} - q^{\alpha N}}{p^\nu - q^\alpha} + 2\gamma \frac{p^{\nu N} - (-1)^N q^{\alpha N}}{p^\nu + q^\alpha} \right. \\ &\quad \left. + \frac{p^{\nu(N+1)} - q^{\alpha(N+1)}}{p^\nu - q^\alpha} + 2\gamma \frac{p^{\nu(N+1)} - (-1)^{N+1} q^{\alpha(N+1)}}{p^\nu + q^\alpha} \right\}. \end{aligned} \quad (5.2)$$

In the basis $\{|n\rangle\}$, it has the diagonal form

$$\mathcal{H}|n\rangle = e_n|n\rangle. \quad (5.3)$$

The energy spectrum, e_n , is given by

$$e_n = \frac{\hbar w}{2} q^\beta \left\{ \frac{p^{\nu n} - q^{\alpha n}}{p^\nu - q^\alpha} + 2\gamma \frac{p^{\nu n} - (-1)^n q^{\alpha n}}{p^\nu + q^\alpha} + \frac{p^{\nu(n+1)} - q^{\alpha(n+1)}}{p^\nu - q^\alpha} + 2\gamma \frac{p^{\nu(n+1)} - (-1)^{n+1} q^{\alpha(n+1)}}{p^\nu + q^\alpha} \right\}. \quad (5.4)$$

As a matter of spectra interpretation, let us introduce a new parametrization:

$$p = \exp(\tau); \quad q = \exp(\rho); \quad \tau\nu = k + \mu; \quad \rho\alpha = k - \mu. \quad (5.5)$$

We readily obtain

$$e_n = \frac{\hbar w}{2} e^{\rho\beta + kn} \left\{ \frac{\sinh(\mu(n+1))}{\sinh(\mu)} + e^{-k} \frac{\sinh(\mu n)}{\sinh(\mu)} + 2\gamma \frac{1 - (-1)^n}{2} \left(\frac{\sinh(\mu(n+1))}{\cosh(\mu)} + e^{-k} \frac{\cosh(\mu n)}{\cosh(\mu)} \right) + 2\gamma \frac{1 + (-1)^n}{2} \left(\frac{\cosh(\mu(n+1))}{\cosh(\mu)} + e^{-k} \frac{\sinh(\mu n)}{\cosh(\mu)} \right) \right\}$$

or

$$e_n = \frac{\hbar w}{2} e^{\rho\beta - k} \left\{ \frac{1 + e^{k+\mu}}{2} \frac{e^{(k+\mu)n}}{\sinh(\mu)} - \frac{1 + e^{k-\mu}}{2} \frac{e^{(k-\mu)n}}{\sinh(\mu)} + 2\gamma \frac{1 - (-1)^n}{2} \left(\frac{1 + e^{k+\mu}}{2} \frac{e^{(k+\mu)n}}{\cosh(\mu)} + \frac{1 - e^{k-\mu}}{2} \frac{e^{(k-\mu)n}}{\cosh(\mu)} \right) + 2\gamma \frac{1 + (-1)^n}{2} \left(\frac{1 + e^{k+\mu}}{2} \frac{e^{(k+\mu)n}}{\cosh(\mu)} - \frac{1 - e^{k-\mu}}{2} \frac{e^{(k-\mu)n}}{\cosh(\mu)} \right) \right\}.$$

It is convenient to separately consider the eigenvalue e_n of \mathcal{H} for n even and odd:

$$e_n = \frac{\hbar w}{2} e^{\rho\beta - k} \left\{ \frac{1 + e^{k+\mu}}{2} \frac{e^{(k+\mu)n}}{\sinh(\mu)} - \frac{1 + e^{k-\mu}}{2} \frac{e^{(k-\mu)n}}{\sinh(\mu)} + 2\gamma \left(\frac{1 + e^{k+\mu}}{2} \frac{e^{(k+\mu)n}}{\cosh(\mu)} + \frac{1 - e^{k-\mu}}{2} \frac{e^{(k-\mu)n}}{\cosh(\mu)} \right) \right\} \quad (5.6)$$

for n odd, and

$$e_n = \frac{\hbar w}{2} e^{\rho\beta - k} \left\{ \frac{1 + e^{k+\mu}}{2} \frac{e^{(k+\mu)n}}{\sinh(\mu)} - \frac{1 + e^{k-\mu}}{2} \frac{e^{(k-\mu)n}}{\sinh(\mu)} + 2\gamma \left(\frac{1 + e^{k+\mu}}{2} \frac{e^{(k+\mu)n}}{\cosh(\mu)} - \frac{1 - e^{k-\mu}}{2} \frac{e^{(k-\mu)n}}{\cosh(\mu)} \right) \right\} \quad (5.7)$$

for n even. From this result, the spectrum of this Hamiltonian is not equidistant and the spacing is given by

$$e_{2n+1} - e_{2n} = \frac{\hbar w}{2} e^{\rho\beta} e^{(2n-1)k} \left(\frac{1}{\sinh(\mu)} + 2\gamma \frac{1}{\cosh(\mu)} \right) \times (e^{2k} \sinh(2(n+1)\mu) - \sinh(2n\mu)). \quad (5.8)$$

According to the above analysis, the spectrum of the Hamiltonian is defined for the parameter $-1 < 2\gamma$ if $\mu > 0$ and $-1 < 2\gamma < -\coth(\mu)$ if $\mu < 0$. In the special case $\mu = 0$, it is reduced to

$$e_n = \frac{\hbar w}{2} e^{\rho\beta+kn} \left\{ (n+\gamma)(1+e^{-k}) + \gamma(-1)^n(1-e^{-k}) + 1 \right\}. \quad (5.9)$$

If, additionally, $k = 0$, $\gamma = 0$, and

$$(\rho = 0, \quad \text{or} \quad \beta = 0, \quad \text{or} \quad \beta \quad (\text{resp.} \quad \rho) \rightarrow -\infty \quad \text{for finite} \quad \rho \quad (\text{resp.} \quad \beta)),$$

then, we recover the spectrum of the ordinary one-dimensional harmonic oscillator:

$$e_n = \hbar w \left(n + \frac{1}{2} \right), \quad (5.10)$$

as expected.

6 Concluding remarks

In this work, we have defined a $(p, q; \alpha, \beta, \nu; \gamma)$ -deformed oscillator algebra which is a straightforward generalization of known deformed algebras. In particular, the Burban multiparameter deformed algebra studied in [12] has been recovered. The deformation structure function and relevant useful formulas have been computed. Then, the corresponding irreducible representations have been classified. Finally, the spectrum of the deformed oscillator Hamiltonian has been investigated and discussed.

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